

ORDINAL INDICES OF SMALL SUBSPACES OF L_p

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ABSTRACT. We calculate ordinal L_p index defined in [3] for Rosenthal's space X_p , ℓ_p and ℓ_2 . We show an infinite dimensional subspace of L_p ($2 < p < \infty$) non isomorphic to ℓ_2 embeds in ℓ_p if and only if its ordinal index is minimum possible. We also give a sufficient condition for a \mathcal{L}_p subspace of $\ell_p \oplus \ell_2$ to be isomorphic to X_p .

1. INTRODUCTION

Kadec and Pelczynski in [7] proved that if X is infinite dimensional subspace of L_p ($2 < p < \infty$) then either X is isomorphic to ℓ_2 or X contains a isomorphic copy of ℓ_p . In addition if X is complemented in L_p ($1 < p < \infty$) and X is not isomorphic to ℓ_2 then it contains a complemented copy of ℓ_p . They also proved that if X is a subspace of L_p ($2 < p < \infty$) such that X is isomorphic to ℓ_2 then X is complemented in L_p . In [5] it was shown that if X is a subspace of L_p ($2 < p < \infty$) such that $\ell_2 \not\hookrightarrow X$ then $X \hookrightarrow \ell_p$. Thus if X is a subspace of L_p ($2 < p < \infty$) such that X is not isomorphic to ℓ_2 and $X \not\hookrightarrow \ell_p$ then X contains an isomorph of $\ell_p \oplus \ell_2$. Moreover if X is \mathcal{L}_p subspace not isomorphic to ℓ_p then X contains a complemented isomorph of $\ell_p \oplus \ell_2$. ℓ_p , ℓ_2 , $\ell_p \oplus \ell_2$ and $\ell_p(\ell_2)$ are referred as small subspaces of L_p ($2 < p < \infty$) and for a long time these were only known examples of complemented subspaces of L_p . In 1970, Rosenthal [8] constructed a complemented subspace of L_p ($1 < p < \infty$), which is denoted by X_p and is not isomorphic to the four spaces mentioned above. X_p is also a kind of small subspace of L_p in the sense that it embeds in $\ell_p \oplus \ell_2$. In [3] the authors defined an ordinal L_p index for separable Banach spaces. With help of this index they proved that there are uncountably many mutually non isomorphic \mathcal{L}_p subspaces of L_p ($1 < p < \infty$). But the exact value of the L_p index of the spaces constructed by them is not known so far.

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It was proved in [6] that for $2 < p < \infty$, if X is a \mathcal{L}_p subspace of $\ell_p \oplus \ell_2$ with unconditional basis then X is isomorphic to one of the spaces ℓ_p , $\ell_p \oplus \ell_2$ or X_p . But for any general \mathcal{L}_p subspace X of $\ell_p \oplus \ell_2$ it is an open question whether X is isomorphic to one of the spaces ℓ_p , $\ell_p \oplus \ell_2$ or X_p (see [4]).

In this work we will first calculate the ordinal L_p index defined in [3] for Rosenthal's space X_p , $\ell_p \oplus \ell_2$, ℓ_p and ℓ_2 . We will show (Corollary 2.8) that an infinite dimensional subspace of L_p ($2 < p < \infty$) non isomorphic to ℓ_2 embeds in ℓ_p if and only if its ordinal index is minimum possible.

It follows that (see Theorem 2.7) for any infinite dimensional subspace of L_p ($2 < p < \infty$), L_p index can have the following three possibilities ω_0 , $\omega_0 \cdot 2$ or greater than equal to ω_0^2 .

Coming back to the question mentioned above, in Theorem 2.9 we will provide a sufficient condition (which is trivially necessary) for a \mathcal{L}_p ($2 < p < \infty$) subspace of $\ell_p \oplus \ell_2$ to be isomorphic to X_p .

We now provide basic background for our work. For notation we closely follow [1] and [3].

Notation: Let X and Y be real Banach spaces. By $X \xrightarrow{(c)} Y$, we mean X is isomorphic to a complemented subspace of Y . $X \equiv Y$ means X is isometric to Y . By $X \stackrel{k}{\sim} Y$, we mean X is isomorphic to Y and there is an isomorphism $S : X \rightarrow Y$ such that $\|S\| \|S^{-1}\| \leq k$. We will write $X \xrightarrow{k} Y$ if $X \stackrel{k}{\sim} Z$ for some isometric subspace Z of Y .

If X is a subspace of L_p by X_0 we denote the subspace of X consisting of mean zero functions only.

For $2 < p < \infty$, we will denote by R_p the constant of equivalence in Rosenthal's inequality [8, Theorem 3]. If in the context p is fixed, we will simply denote it by R .

We now recall the notion of independent sum and R_p^α spaces.

Consider the sequence $\{X_n\}$ of Banach spaces where X_n is subspace of $L_p(\Omega_n, \mu_n)$ for some probability measure μ_n . Let $\mu = \prod \mu_n$ be the product measure on $\Omega = \prod \Omega_n$. For each $n \in \mathbb{N}$, we denote the canonical projection from Ω to Ω_n by P_n and $j_n(f) = f \circ P_n$. Let X be the subspace of $L_p(\Omega, \mu)$ consisting of constant functions and j be the inclusion of X into $L_p(\Omega, \mu)$. By $(\sum X_n)_{Ind,p}$ we denote the closed linear span of $\cup j_n(X_n) \cup j(X)$ in $L_p(\Omega, \mu)$.

In defining R_p^α spaces we follow the view point considered in [1].

For $1 \leq p < \infty$, let $R_p^0 = L_p^0 = [1]_{L_p}$. Now suppose that R_p^n has been defined for $n \in \mathbb{N}$. We define $R_p^{n+1} = \ell_p^2 \otimes R_p^n$ and $R_p^{\omega_0} = (\sum R_p^n)_{Ind,p}$. Thus

we have $R_p^{\omega_0} = (\Sigma \ell_p^{2^n})_{Ind,p}$. In general for any ordinal $\alpha < \omega_1$, we define $R_p^{\alpha+1} = \ell_p^2 \otimes R_p^\alpha$ and for a limit ordinal β we put $R_p^\beta = (\Sigma_{\alpha < \beta} R_p^\alpha)_{Ind,p}$. It is known that Rosenthal's space X_p is isomorphic to $R_p^{\omega_0}$ for $1 < p < \infty$.

Now we will define the notion of $(p, 2, (w_n))$ sum of subspaces of L_p given in [1].

Definition 1.1. Let (X_n) be a sequence of subspaces of $L_p(\Omega, \mu)$ for some probability measure μ and $\{w_n\}$ be a sequence of real numbers, $0 \leq w_n \leq 1$. For any sequence (x_n) such that $x_n \in X_n$, let

$$\|(x_n)\|_{p,2,(w_n)} = \max\{(\sum \|x_n\|_p^p)^{1/p}, (\sum \|x_n\|_2^2 w_n^2)^{1/2}\}$$

and

$$X = (\Sigma X_n)_{p,2,(w_n)} = \{(x_n) : x_n \in X_n \text{ for all } n \text{ and } \|(x_n)\|_{p,2,(w_n)} < \infty\}.$$

Remark 1.2. If $w_n = 1$ for all $n \in \mathbb{N}$ then we will denote $(\Sigma X_n)_{p,2,(w_n)}$ by $(\Sigma X_n)_{p,2,(1)}$. From above definition we can easily verify that if X is a subspace of L_p then $(\Sigma X)_{p,2,(1)}$ is stable under taking $(p, 2, (1))$ sum. We will denote $(\Sigma X)_{p,2,(1)}$ by $(X)_{p,2,(1)}$. Thus $X \sim (X)_{p,2,(1)}$ if and only if X is stable under taking $(p, 2, (1))$ sum.

Remark 1.3. In [1, Lemma 2.1] it was proved that if X_n is a subspace of $L_p(\Omega_n, \mu_n)$ for each $n \in \mathbb{N}$ then $(\Sigma X_n)_{Ind,p}$ is isomorphic to $(\Sigma X_{n,0})_{Ind,p} \oplus L_p^0$. Thus it follows from Rosenthal's inequality and Remark 1.2 that for each limit ordinal $\omega_0 \leq \alpha < \omega_1$, R_p^α ($2 < p < \infty$) is stable under taking $(p, 2, (1))$ sum. It is known that (see [1, Corollary 2.10]) R_p^α ($2 < p < \infty$) spaces are isomorphically distinct at limit ordinals. Thus for each ordinal $\omega_0 \leq \alpha < \omega_1$, R_p^α ($2 < p < \infty$) is stable under taking $(p, 2, (1))$ sum.

We will use the following result, proof of which is essentially contained in the proof of [1, Theorem 2.4].

Theorem 1.4. Let $2 < p < \infty$. There exists a constant A such that $R_p^{\omega_0 \cdot 2} \overset{A}{\sim} (\Sigma R_{p,0}^{\omega_0+n})_{p,2,(1)}$.

We now recall the definition of ordinal L_p index for separable Banach spaces from [3].

Definition 1.5. For $n \in \mathbb{N} \cup \{0\}$, let D_n be the set of all n -strings of 0 's and 1 's. For a separable Banach space X , X^{D_n} be the set of all functions from D_n to X . Let $X^{\mathcal{D}} = \cup_{n=0}^{\infty} X^{D_n}$.

For $u \in X^{\mathcal{D}}$ we write $|u| = n$ if $u \in X^{D_n}$. For $t \in D_n$ and $s \in D_m$ we denote by $t \cdot s$ element in D_{n+m} given by $t_1 \cdots t_n \cdot s_1 \cdots s_m$.

On $X^{\mathcal{D}}$ we define a strict partial order by $u \prec v$ if $|u| < |v|$ and for $k = |v| - |u|$, $u(t) = 2^{-k/p} \sum_{s \in D_k} v(t \cdot s)$.

Let $1 \leq p < \infty$ and $0 < \delta \leq 1$. Let \overline{X}^δ be the set of all $u \in X^{\mathcal{D}}$ such that

$$(1) \quad \delta \left(\sum_{t \in D_{|u|}} |c(t)|^p \right)^{1/p} \leq \left\| \sum_{t \in D_{|u|}} c(t)u(t) \right\|_X \leq \left(\sum_{t \in D_{|u|}} |c(t)|^p \right)^{1/p}$$

for all $c \in \mathbb{R}^{D_{|u|}}$.

Remark 1.6. As a consequence of (1) we observe that if $u = (u_1, \dots, u_{2^k}) \in \overline{X}^\delta$ then $\{u_1, \dots, u_{2^k}\}$ are linearly independent. In case of $\delta = 1$, $\{u_1, \dots, u_{2^k}\}$ spans $\ell_p^{2^k}$ isometrically.

Let $H_0^\delta(X) = \overline{X}^\delta$. If $\alpha = \beta + 1$ and $H_\beta^\delta(X)$ has been defined, then take

$$H_\alpha^\delta(X) = \{u \in H_\beta^\delta(X) : u \prec v \text{ for some } v \in H_\beta^\delta(X)\}.$$

If α is a limit ordinal we define $H_\alpha^\delta(X) = \cap_{\beta < \alpha} H_\beta^\delta(X)$.

In [3] it was proved that for $1 \leq p < \infty$ if $L_p \not\hookrightarrow X$ then for any $0 < \delta \leq 1$ there exists an ordinal $\beta < \omega_1$ such that $H_\beta^\delta(X) = H_{\beta+1}^\delta(X)$. Let $h_p(\delta, X)$ be the least ordinal α such that $H_\alpha^\delta(X) = H_{\alpha+1}^\delta(X)$. If $L_p \not\hookrightarrow X$, we define $h_p(X) = \sup_{0 < \delta \leq 1} h_p(\delta, X)$. If $L_p \hookrightarrow X$ by convention we take $h_p(X) = \omega_1$. In [3] it was proved that for $1 \leq p < \infty$, $h_p(X) < \omega_1$ if and only if $L_p \not\hookrightarrow X$.

We will be using the following results repeatedly while calculating the L_p index of $R_p^{\omega_0}$, ℓ_p and ℓ_2 .

Theorem 1.7. [3, Theorem 2.1] If X and Y are two separable Banach spaces such that $X \hookrightarrow Y$ then $h_p(X) \leq h_p(Y)$.

Theorem 1.8. [3, Theorem 2.4] Let $1 \leq p < \infty$, $0 \leq \alpha < \omega_1$. Then $1 \in H_\alpha^1(R_p^\alpha)$.

FACT [3, Lemma 2.5]: Let X be a separable Banach, $0 < \delta \leq 1$ and $\alpha < \omega_1$. Let $e \in H_\alpha^\delta(X)$. Let \bar{e} be the element of $(X \oplus X)_p^{\mathcal{D}}$ defined by $\bar{e}(t) = 2^{-\frac{1}{p}}(e(t) \oplus e(t))$ for all $t \in D_{|e|}$. Then $\bar{e} \in H_{\alpha+1}^\delta((X \oplus X)_p)$. As a consequence if X is isomorphic to its square and $h_p(X) > \alpha$, for some limit ordinal α then $h_p(X) \geq \alpha + \omega_0$.

Remark 1.9. Let $1 \leq p < \infty$. It is easy to observe that $h_p(\ell_p^{2^n}) = n + 1$. Hence by Theorem 1.7 for any infinite dimensional space X , we have $h_p(X) \geq \omega_0$.

Remark 1.10. (a) It was proved by Rosenthal in [8] that for $1 < p < \infty$, $\ell_p(\ell_2) \not\hookrightarrow X_p$ (which is isomorphic to $R_p^{\omega_0}$).

(b) For a sequence of scalars $\{w_n\}$, $0 \leq w_n \leq 1$ and $2 < p < \infty$ we recall the Rosenthal's condition which is: for each $\epsilon > 0$, $\sum_{w_n < \epsilon} w_n^{\frac{2p}{p-2}} = \infty$.

If the sequence $\{w_n\}$ satisfies Rosenthal's condition then it was proved in [1, Theorem 2.4] that $R_p^{\omega_0 \cdot 2} \sim (R_p^{\omega_0})_{p,2,(w_n)}$. It follows from [1, Proposition 2.11] that by breaking the collection $\{w_n\}$ into three disjoint sub-collections namely $\{w_n^1\}$, $\{w_n^2\}$ and $\{w_n^3\}$ satisfying the conditions; $\{w_n^1\}$ satisfies Rosenthal's condition, $\inf w_n^2 > 0$ and $\sum (w_n^3)^{\frac{2p}{p-2}} < \infty$, we get

$$R_p^{\omega_0 \cdot 2} \sim (R_p^{\omega_0})_{p,2,(w_n^1)} \oplus (R_p^{\omega_0})_{p,2,(1)} \oplus \ell_p(R_p^{\omega_0}).$$

Since $\ell_2 \hookrightarrow R_p^{\omega_0}$, we get $\ell_p(\ell_2) \hookrightarrow R_p^{\omega_0 \cdot 2}$.

(c) It follows from (a) and (b) above that $R_p^{\omega_0 \cdot 2} \not\hookrightarrow R_p^{\omega_0}$.

2. MAIN RESULTS

The following Lemma is key to calculate L_p index of ℓ_p , ℓ_2 and $R_p^{\omega_0}$.

Lemma 2.1. Let $2 < p < \infty$ and X be a subspace of L_p . If for some $0 < \delta \leq 1$ and $n \in \mathbb{N}$, $H_{\omega_0+n}^\delta(X) \neq \emptyset$ then there exists a constant C (depending on δ , p and X only) such that $R_{p,0}^{\omega_0+n} \xrightarrow{C} (X)_{p,2,(1)}$.

Proof. Without loss of generality we assume that X consists of only mean zero functions (otherwise we write $X = X_0 \oplus L_p^0$ and work with X_0). Let $u_n \in H_{\omega_0+n}^\delta(X)$, $v_n \in H_{\omega_0}^\delta(X)$ such that $u_n \prec v_n$. So we have $|v_n| \geq n$. For all $k \in \mathbb{N}$ we can find some $v_k^n \in H_k^\delta(X)$ such that $v_n \prec v_k^n$. Further for all k we can find some $w_k^n \in H_0^\delta(X)$ such that $v_k^n \prec w_k^n$ and $|w_k^n| \geq n + k + 1$.

Let $|v_n| = m$. Then $m \geq n$ and $|w_k^n| \geq m + k + 1$. For a fixed string $t_1 \cdots t_m$ of 0's and 1's we have

$$(2) \quad v_n(t_1 \cdots t_m) = 2^{\frac{-(|w_k^n| - m)}{p}} \sum_{s \in D_{|w_k^n| - m}} w_k^n(t_1 \cdots t_m \cdot s).$$

Let $W_k^n(t_1 \cdots t_m)$ be the subspace spanned by components of w_k^n which appear in the representation of $v_n(t_1 \cdots t_m)$ in (2) above. It is immediate to observe that $\ell_p^{2^k} \xrightarrow{\frac{1}{\delta}} W_k^n(t_1 \cdots t_m)$ and this copy consists of mean zero functions only (see Remark 1.6). Let $X_{t_1 \cdots t_m} = (W_k^n(t_1 \cdots t_m))_{Ind,p}$, then $(\ell_p^{2^k})_{Ind,p} \xrightarrow{\frac{1}{\delta}} X_{t_1 \cdots t_m}$, that is

$$(3) \quad R_{p,0}^{\omega_0} \xrightarrow{\frac{1}{\delta}} X_{t_1 \cdots t_m}.$$

By taking $(p, 2, (1))$ sum on both sides of (3) we have $(\sum_1^{2^m} R_{p,0}^{\omega_0})_{p,2,(1)} \xrightarrow{\frac{1}{\delta}}$
 $((X)_{Ind,p})_{p,2,(1)}$. It follows from Rosenthal's inequality that $(X)_{Ind,p} \stackrel{R}{\sim}$
 $(X)_{p,2,(1)}$ and $(\sum_1^{2^m} R_{p,0}^{\omega_0})_{Ind,p} \stackrel{R}{\sim} (\sum_1^{2^m} R_{p,0}^{\omega_0})_{p,2,(1)}$. Since $R_{p,0}^{\omega_0+m} \equiv (\sum_1^{2^m} R_{p,0}^{\omega_0})_{Ind,p}$
 so we have $R_{p,0}^{\omega_0+m} \xrightarrow{\frac{R^2}{\delta}} ((X)_{p,2,(1)})_{p,2,(1)}$. Thus we have $R_{p,0}^{\omega_0+m} \xrightarrow{\frac{R^2 B}{\delta}} (X)_{p,2,(1)}$
 where B is the constant such that $(X)_{p,2,(1)} \stackrel{B}{\sim} ((X)_{p,2,(1)})_{p,2,(1)}$. Since $m \geq n$
 so we get the desired result. \square

We will use the following result for calculating the L_p index of ℓ_p and ℓ_2 which we will prove later.

Theorem 2.2. *Let $2 < p < \infty$. Then $h_p(R_p^{\omega_0}) = \omega_0 \cdot 2$.*

We believe the following Lemma is essentially known, however we include the proof for completion. Let $\{e_n\}$ denotes the standard unit vector basis of ℓ_p .

Lemma 2.3. *Let $1 \leq p < \infty$, $u = (u_1, \dots, u_{2^k}) \in H_0^1(\ell_p)$ and $u_i = \sum_{j \in N_i} b_j^i e_j$, where $N_i \subseteq \mathbb{N}$, $1 \leq i \leq 2^k$. Then for $i \neq j$, $1 \leq i, j \leq 2^k$ we have $N_i \cap N_j = \emptyset$.*

Proof. If possible let $N_1 \cap N_2 \neq \emptyset$ and $n_0 \in N_1 \cap N_2$. Since $u \in H_0^1(\ell_p)$, we have $\|c_1 u_1 + c_2 u_2\|_p^p = |c_1|^p + |c_2|^p$, $\sum_{j \in N_i} |b_j^i|^p = 1$ for all $c_1, c_2 \in \mathbb{R}$ and $1 \leq i \leq 2^k$. Then we have

$$\|c_1 u_1 + c_2 u_2\|_p^p \leq |c_1 b_{n_0}^1 + c_2 b_{n_0}^2|^p + |c_1|^p \sum_{i \neq n_0} |b_i^1|^p + |c_2|^p \sum_{i \neq n_0} |b_i^2|^p.$$

Taking $c_1 = b_{n_0}^2$ and $c_2 = -b_{n_0}^1$, we get

$$\begin{aligned} \|c_1 u_1 + c_2 u_2\|_p^p &\leq |c_1|^p \sum_{i \neq n_0} |b_i^1|^p + |c_2|^p \sum_{i \neq n_0} |b_i^2|^p \\ &= |c_1|^p (1 - |b_{n_0}^1|^p) + |c_2|^p (1 - |b_{n_0}^2|^p) \\ &< |c_1|^p + |c_2|^p. \end{aligned}$$

This contradicts that $\|c_1 u_1 + c_2 u_2\|_p^p = |c_1|^p + |c_2|^p$.

Hence $N_1 \cap N_2 = \emptyset$. Similarly we can show that $N_i \cap N_j = \emptyset$ for $i \neq j$, $1 \leq i, j \leq 2^k$. \square

Lemma 2.4. *Let $1 \leq p < \infty$. Then $h_p(1, \ell_p) = \omega_0$.*

Proof. Let $u = (u_1, \dots, u_{2^k}) \in H_0^1(\ell_p)$ and $u_i = \sum_{j \in N_i} a_j^i e_j$, where N_i is a subset of \mathbb{N} , $1 \leq i \leq 2^k$. If there doesn't exist any $N_u < \omega_0$ such that $u \notin H_{N_u}^1(\ell_p)$, we can find $v_n \in H_0^1(\ell_p)$ with $|v_n| \uparrow \omega_0$ and $u \prec v_n$ for each n . Thus for each i we can find some fixed k string $t_1 \cdots t_k$ of 0's and 1's such that $u_i = 2^{\frac{-(|v_n| - k)}{p}} \sum_{s \in D_{|v_n| - k}} v_n(t_1 \cdots t_k \cdot s)$. Using Lemma 2.3 we have for $s \in D_{|v_n| - k}$ and any k string $t_1 \cdots t_k$ of 0's and 1's, $v_n(t_1 \cdots t_k \cdot s) = \sum_{j \in N_s} c_j^n e_j$, where $\cup_{s \in D_{|v_n| - k}} N_s = N_i$ and $N_{s_1} \cap N_{s_2} = \emptyset$ if $s_1 \neq s_2$. Thus for each $j \in N_i$ and $n \in \mathbb{N}$ can find some j_0 such that $a_j^i = \frac{c_{j_0}^n}{2^{\frac{|v_n| - k}{p}}}$.

Since for all n and $j \in N_i$ we have $|c_j^n| \leq 1$ we conclude that $a_j^i = 0$ for all $j \in N_i$, which contradicts $\|u_i\| = 1$, $1 \leq i \leq 2^k$. Thus there exists some $N_u < \omega_0$ such that $u \notin H_{N_u}^1(\ell_p)$. Hence $h_p(1, \ell_p) \leq \omega_0$. But it is easy to observe that $h_p(1, \ell_p^{2^n}) = n + 1$. Thus we have $h_p(1, \ell_p) = \omega_0$. \square

Proposition 2.5. *Let $2 < p < \infty$. Then $h_p(\ell_p) = \omega_0$.*

Proof. It follows from Lemma 2.4 that $h_p(\ell_p) \geq \omega_0$. If $h_p(\ell_p) > \omega_0$ then by using FACT we have $h_p(\ell_p) \geq \omega_0 \cdot 2$. We know that $\ell_p \hookrightarrow R_p^{\omega_0}$ thus Theorem 2.2 and Theorem 1.7 implies that $h_p(\ell_p) = \omega_0 \cdot 2$. Thus we can find some $0 < \delta \leq 1$ such that $H_{\omega_0}^\delta(\ell_p) \neq \emptyset$. Then again by FACT we have $H_{\omega_0+1}^\delta((\ell_p \oplus \ell_p)_p) \neq \emptyset$. Since $(\ell_p \oplus \ell_p)_p \equiv \ell_p$, we have $H_{\omega_0+1}^\delta(\ell_p) \neq \emptyset$. By similar arguments we can show that $H_{\omega_0+n}^\delta(\ell_p) \neq \emptyset$ for all $n \in \mathbb{N}$. Thus by Lemma 2.1 there exists some constant C such that $R_{p,0}^{\omega_0+n} \xrightarrow{C} (\ell_p)_{p,2,(1)}$ for all $n \in \mathbb{N}$. By taking $(p, 2, (1))$ sum on both sides we have $(\sum R_{p,0}^{\omega_0+n})_{p,2,(1)} \xrightarrow{C} ((\ell_p)_{p,2,(1)})_{p,2,(1)}$. Now it follows from Theorem 1.4 that $R_p^{\omega_0 \cdot 2} \xrightarrow{CA} ((\ell_p)_{p,2,(1)})_{p,2,(1)}$. But again by Remark 1.3 there exists some constant B such that $(R_p^{\omega_0})_{p,2,(1)} \xrightarrow{B} R_p^{\omega_0}$. Also we have $((\ell_p)_{p,2,(1)})_{p,2,(1)} \xrightarrow{(c)} ((R_p^{\omega_0})_{p,2,(1)})_{p,2,(1)} \xrightarrow{B^2} R_p^{\omega_0}$. Thus $R_p^{\omega_0 \cdot 2} \xrightarrow{CAB^2} R_p^{\omega_0}$. Which is a contradiction to Remark 1.10. Thus $h_p(\ell_p) = \omega_0$. \square

Now we will calculate the ordinal L_p index of ℓ_2 .

Proposition 2.6. *Let $2 < p < \infty$. Then $h_p(\ell_2) = \omega_0$.*

Proof. It follows from Remark 1.9 that $h_p(\ell_2) \geq \omega_0$. If we assume $h_p(\ell_2) > \omega_0$ then by FACT it follows that $h_p(\ell_2) \geq \omega_0 \cdot 2$. Since $\ell_2 \hookrightarrow R_p^{\omega_0}$, by Theorem 2.2 and Theorem 1.7 we have that $h_p(\ell_2) = \omega_0 \cdot 2$. Thus we can find some $0 < \delta \leq 1$ such that $H_{\omega_0+1}^\delta(\ell_2) \neq \emptyset$. By using Lemma 2.1 we have $R_{p,0}^{\omega_0} \hookrightarrow (\ell_2)_{p,2,(1)}$. Remark 1.3 and the fact $\ell_2 \sim (\mathbb{R})_{p,2,(1)}$ implies that ℓ_2 is stable under taking

$(p, 2, (1))$ sum. Thus we conclude $R_p^{\omega_0} \hookrightarrow \ell_2$, which is not possible as $\ell_p \hookrightarrow R_p^{\omega_0}$. This shows $h_p(\ell_2) = \omega_0$. \square

For $2 < p < \infty$, let X be infinite dimensional subspace of L_p . In the following Theorem we show that value of L_p index of X can have only three possible choices.

Theorem 2.7. *Let $2 < p < \infty$ and X be an infinite dimensional subspace of L_p . Then either $h_p(X) = \omega_0$, $h_p(X) = \omega_0 \cdot 2$ or $h_p(X) \geq \omega_0^2$.*

Proof. We recall from [4] that if X is a subspace of L_p then either $X \hookrightarrow \ell_p \oplus \ell_2$ or $\ell_p(\ell_2) \hookrightarrow X$. Suppose $X \hookrightarrow \ell_p \oplus \ell_2$. Note that $R_p^{\omega_0}$ and $\ell_p \oplus \ell_2$ embeds in each other (see [8]). Thus we have from Theorem 2.2 and Theorem 1.7 that $h_p(X) \leq \omega_0 \cdot 2$. If $X \hookrightarrow \ell_p$ or $X \sim \ell_2$ then by Proposition 2.5 and Proposition 2.6 we have $h_p(X) = \omega_0$. In otherwise $\ell_p \oplus \ell_2 \hookrightarrow X$ hence $h_p(X) = \omega_0 \cdot 2$.

In the case when $\ell_p(\ell_2) \hookrightarrow X$, we actually have $R_p^{\omega_0 \cdot n} \hookrightarrow X$ for each $n \in \mathbb{N}$. This is because $R_p^{\omega_0 \cdot n} \hookrightarrow \ell_p(\ell_2)$. (This fact is highly non trivial. In [9] Schechtman constructed countably many mutually non isomorphic subspaces of $\ell_p(\ell_2)$, by taking repeated tensor product of X_p , denoted by $\otimes_1^n X_p$. These spaces are \mathcal{L}_p spaces. Alspach proved in [1] that $R_p^{\omega_0 \cdot n} \hookrightarrow \otimes_1^n X_p$.)

It was proved in [3] that for any ordinal $\alpha < \omega_1$, $h_p(R_p^\alpha) \geq \alpha + 1$. Hence we have $h_p(X) \geq \omega_0 \cdot n + 1$ for all $n \in \mathbb{N}$. This shows that $h_p(X) \geq \omega_0^2$. \square

We now prove Theorem 2.2.

Proof of Theorem 2.2: It is known that (see [1, Corollary 2.10]) R_p^α spaces are isomorphically distinct at limit ordinals. Thus we have $R_p^{\omega_0+k} \sim R_p^{\omega_0}$ for all $k \in \mathbb{N}$ and using Theorem 1.8 we have $h_p(R_p^{\omega_0}) \geq \omega_0 \cdot 2$. We will show that $h_p(R_p^{\omega_0}) = \omega_0 \cdot 2$. Suppose on the contrary $h_p(R_p^{\omega_0}) > \omega_0 \cdot 2$. Hence there exists a $0 < \delta \leq 1$ such that $h_p(\delta, R_p^{\omega_0}) > \omega_0 \cdot 2$. Thus for each $n \in \mathbb{N}$, $H_{\omega_0+n}^\delta(R_p^{\omega_0}) \neq \emptyset$. By using Lemma 2.1 there exists some constant C such that for all $n \in \mathbb{N}$ we have $R_{p,0}^{\omega_0+n} \xrightarrow{C} (R_p^{\omega_0})_{p,2,(1)}$. By Taking $(p, 2, (1))$ sum we have $(\sum R_{p,0}^{\omega_0+n})_{p,2,(1)} \xrightarrow{C} ((R_p^{\omega_0})_{p,2,(1)})_{p,2,(1)}$. Now by using Theorem 1.4 and Remark 1.3 we have $R_p^{\omega_0 \cdot 2} \overset{A}{\sim} (\sum R_{p,0}^{\omega_0+n})_{p,2,(1)}$ and $(R_p^{\omega_0})_{p,2,(1)} \overset{B}{\sim} R_p^{\omega_0}$ for some constant B . From this we conclude that $R_p^{\omega_0 \cdot 2} \xrightarrow{CAB^2} R_p^{\omega_0}$, which contradicts Remark 1.10. Thus we have $h_p(R_p^{\omega_0}) = \omega_0 \cdot 2$.

Corollary 2.8. *Let X be an infinite dimensional subspace of L_p ($2 < p < \infty$) such that $X \not\sim \ell_2$. Then $h_p(X) = \omega_0$ if and only if $X \hookrightarrow \ell_p$.*

Proof. If $X \hookrightarrow \ell_p$ then by Proposition 2.5, Theorem 1.7 and Remark 1.9 we have $h_p(X) = \omega_0$. Conversely if $X \not\sim \ell_2$ and $X \not\hookrightarrow \ell_p$ then $\ell_p \oplus \ell_2 \hookrightarrow X$. Since $\ell_p \oplus \ell_2$ and $R_p^{\omega_0}$ embeds in each other thus by Theorem 1.7 and Theorem 2.2 we have $h_p(X) \geq \omega_0 \cdot 2$. \square

To end this note we will provide a sufficient condition (which is trivially necessary) for a \mathcal{L}_p subspace of $\ell_p \oplus \ell_2$ to be isomorphic to X_p .

Theorem 2.9. *Let X be a \mathcal{L}_p subspace of $\ell_p \oplus \ell_2$, $2 < p < \infty$ such that $X \not\hookrightarrow \ell_p$. Then $X \sim X_p$ if and only if X is stable under taking $(p, 2, (1))$ sum.*

Proof. If X is a \mathcal{L}_p subspace of $\ell_p \oplus \ell_2$ such that $X \not\hookrightarrow \ell_p$ then $\ell_p \oplus \ell_2 \xrightarrow{(c)} X$. By taking $(p, 2, (1))$ sum on both sides we get $(\ell_p \oplus \ell_2)_{p,2,(1)} \hookrightarrow (X)_{p,2,(1)}$ and by [1, Lemma 2.5] this copy is complemented. Thus $(\ell_p \oplus \ell_2)_{p,2,(1)} \xrightarrow{(c)} (X)_{p,2,(1)}$. We claim that $(\ell_p \oplus \ell_2)_{p,2,(1)} \sim R_p^{\omega_0}$. To see this first note that $\ell_p^{2^n} \xrightarrow{(c)} \ell_p$ and for each n the projection constant is 1. Thus we have $(\sum \ell_p^{2^n})_{p,2,(1)} \xrightarrow{(c)} (\ell_p)_{p,2,(1)}$. Since $R_p^{\omega_0} \sim (\sum \ell_p^{2^n})_{p,2,(1)}$ there exists an isomorphic copy of $R_p^{\omega_0}$ complemented in $(\ell_p)_{p,2,(1)}$. Observe that this copy is stable under taking $(p, 2, (1))$ sum. Since ℓ_p is complemented in this copy we have $(\ell_p)_{p,2,(1)}$ is complemented there. As both concerned spaces are isomorphic to their square by decomposition method we have $(\ell_p)_{p,2,(1)} \sim R_p^{\omega_0}$. Now $(\ell_p \oplus \ell_2)_{p,2,(1)} \sim (\ell_p)_{p,2,(1)} \oplus (\ell_2)_{p,2,(1)} \sim R_p^{\omega_0} \oplus \ell_2 \sim R_p^{\omega_0}$. Coming back to the proof, by the claim we have X contains a complemented copy of $R_p^{\omega_0}$. Since X is a \mathcal{L}_p subspace of $\ell_p \oplus \ell_2$ so by [4, Proposition 8.6] we have $X \xrightarrow{(c)} X_p$. Recall that $R_p^{\omega_0} \sim X_p$. Since X is stable under taking $(p, 2, (1))$ sum it is isomorphic to its square. Using decomposition method we have $X \sim X_p$. \square

Remark 2.10. *It follows a complemented subspace X of X_p is stable under taking $(p, 2, (1))$ sum if and only if X satisfies condition (2) of [2, Theorem 2.1].*

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